Calculus 2 Notes (2023/2024)

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Contents

1 Vectors

Standard basis vectors: $\boldsymbol{i} = \langle 1,0,0\rangle \quad \boldsymbol{j} = \langle 0,1,0\rangle \quad \boldsymbol{k} = \langle 0,0,1\rangle$

Unit vector of a vector $\boldsymbol{a} \neq \boldsymbol{0}$: $\quad \boldsymbol{u} = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}$

1.1 Dot product

Definition *Dot product*

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + \ldots + a_n b_n$

$$\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2 \qquad \boldsymbol{a}\cdot\boldsymbol{0} = 0 \qquad \boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a} \qquad \boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c} \qquad (c\boldsymbol{a})\cdot\boldsymbol{b} = c(\boldsymbol{a}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot(c\boldsymbol{b})$$

Theorem

If θ is the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\quad \boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta \quad$ and $\quad \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|} = \cos\theta$

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **orthogonal** (or **perpendicular**) if and only if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$

$$\boldsymbol{a}\cdot\boldsymbol{b} < 0 \implies \theta > \frac{\pi}{2} \qquad \boldsymbol{a}\cdot\boldsymbol{b} = 0 \implies \theta = \frac{\pi}{2} \qquad \boldsymbol{a}\cdot\boldsymbol{b} > 0 \implies \theta < \frac{\pi}{2}$$

The **direction angles** α, β, γ are the angles that a vector makes with the x, y and z axes.

Direction cosines: $\cos\alpha = \dfrac{a_1}{|\boldsymbol{a}|} \qquad \cos\beta = \dfrac{a_2}{|\boldsymbol{a}|} \qquad \cos\gamma = \dfrac{a_3}{|\boldsymbol{a}|} \qquad \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle$

Definition *Projections*

The **scalar projection** of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\quad \text{comp}_{\boldsymbol{a}}\,\boldsymbol{b} = |\boldsymbol{b}|\cos\theta = \dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}\cdot\boldsymbol{b}$

The **vector projection** of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is $\quad \text{proj}_{\boldsymbol{a}}\,\boldsymbol{b} = \left(\dfrac{\boldsymbol{a}}{|\boldsymbol{a}|}\right)^2\cdot\boldsymbol{b}$

1.2 Cross product

Definition *Cross product*

If $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3\rangle$, then $\boldsymbol{a}\times\boldsymbol{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$

Matrix representation of the cross product: $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

The cross product is neither commutative nor associative.

Theorem

The vector $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

Theorem

If θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ $(0 \leq \theta \leq \pi)$, then $|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$

Two nonzero vectors $\boldsymbol{a}$ are parallel if and only if $\boldsymbol{a}\times\boldsymbol{b} = \boldsymbol{0}$

$$\boldsymbol{i}\times\boldsymbol{j} = \boldsymbol{k} \qquad \boldsymbol{j}\times\boldsymbol{k} = \boldsymbol{i} \qquad \boldsymbol{k}\times\boldsymbol{i} = \boldsymbol{j} \qquad \boldsymbol{j}\times\boldsymbol{i} = -\boldsymbol{k} \qquad \boldsymbol{k}\times\boldsymbol{j} = -\boldsymbol{i} \qquad \boldsymbol{i}\times\boldsymbol{k} = -\boldsymbol{j}$$

Theorem

1. $\boldsymbol{a}\times\boldsymbol{b} = -\boldsymbol{b}\times\boldsymbol{a}$
2. $(c\boldsymbol{a})\times\boldsymbol{b} = c(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}\times(c\boldsymbol{b})$
3. $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}$
4. $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$
5. $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$
6. $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$

Triple product: $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

The volume of the parallelepiped determined by the vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is $\quad V = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$

If the triple product of three vectors is 0, they must be in the same plane. (they are **coplanar**)

1.3 Lines and planes

Vector equation of a line
Let $\boldsymbol{r}_0$ and $\boldsymbol{r}$ be the position vectors of a fixed point P_0 and an arbitrary point P in a line L and let $\boldsymbol{v}$ be a vector parallel to $\boldsymbol{r}_0 - \boldsymbol{r}$. Then $\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{v}$ is contained in L for all t.

The **line segment** from $\boldsymbol{r}_0$ to $\boldsymbol{r}_1$ is given by the vector equation $\boldsymbol{r}(t) = (1-t)\boldsymbol{r}_0 + t\boldsymbol{r}_1 \quad 0 \leq t \leq 1$

A **plane** in space is determined by a point p_0 and a **normal vector** $\boldsymbol{n}$.
Two planes are **parallel** if their normal vectors are parallel.
Let P be an arbitrary point on the plane. Then $\boldsymbol{n}$ is orthogonal to the vector $\boldsymbol{p} - \boldsymbol{p_0}$.

Vector equation of a plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
where $\langle a, b, c\rangle$ is the normal vector and $\langle x_0, y_0, z_0\rangle$ are the coordinates of p_0

1.4 Surfaces

Definition *Quadric surface*

A **quadric surface** is a surface with an equation of the form
$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$

The equation of a quadric surface can be brought into one of the standard forms:
$Ax^2 + By^2 + Cz^2 + J = 0 \quad$ or $\quad Ax^2 + By^2 + Iz = 0$

A **cylinder** is a curve (on a plane) extended into the space perpendicular to the plane.

A **trace** is the intersection of a surface and a plane.
An **ellipsoid** is a surface where all traces are ellipses. $\left(\frac{x^2}{a^2} + \frac{x^2}{y^2} + \frac{z^2}{c^2} = 1 \text{ with } a, b, c \text{ positive}\right)$

How to sketch surfaces:

- Form traces by fixing x, y or z at some $k \in \mathbb{R}$ (preferably 0)
- Extend traces in $\mathbb{R}^3$

2 Vector functions

A **vector function** $r(t) = \langle r_1(t), \ldots, r_n(t)\rangle$ is a function $I \to \mathbb{R}^n$ where I is an interval.

The distance between points $u, v \in \mathbb{R}^n$ is $\|u - v\|$

2.1 Limits

Definition *Limit of a vector function*

$L \in \mathbb{R}^n$ is the **limit** of $r : I \to \mathbb{R}^n$ if $\quad \forall \varepsilon > 0 \;\; \exists \delta > 0 \;$ s.t. $\; |t - a| < \delta \implies \|r(t) - L\| < \varepsilon$

Lemma

Let $r(t) = \langle f(t), g(t), h(t)\rangle$ and $L = \langle L_1, L_2, L_3\rangle \in \mathbb{R}^3$.

$$L \text{ is a limit of } r \text{ at } a \iff \begin{array}{l} f \text{ has limit } L_1 \text{ at } a \\ g \text{ has limit } L_2 \text{ at } a \\ h \text{ has limit } L_3 \text{ at } a \end{array}$$

A similar statement also holds for $\mathbb{R}^n$.

Definition *Continuity of vector functions*

r is **continuous** at $a \in I$ if $\lim_{t \to a} r(t) = r(a)$

Definition *Space curve*

Let $I \in \mathbb{R}$ and $r : I \to \mathbb{R}^3$ continuous.
Then $C = r(I) = \{t \in I : (f(t), g(t), h(t)) \in \mathbb{R}^3\}$ is a **space curve**.

2.2 Derivatives

Definition *Differentiability of vector functions*

$r : I \to \mathbb{R}^n$ is **differentiable** at $t \in I$ if the limit $r'(t) = \lim_{h \to 0} \dfrac{r(t+h) - r(t)}{h}$ exists.

$r'(t)$ is a vector in $\mathbb{R}^n$ which is tangent to the curve at t.
If $r(t)$ is the position of an object and t is time, then $r'(t)$ is its velocity and $\|r'(t)\|$ is its speed.
The derivative of r depends on the parametrization, not just the curve.

Theorem

A vector function is differentiable if and only if its component functions are differentiable.

Definition *Unit tangent vector*

If $r'(t) \neq 0$ then $T(t) = \dfrac{r'(t)}{\|r'(t)\|}$

Theorem *Differentiation rules*

Let $u : I \to \mathbb{R}^n$, $v : I \to \mathbb{R}^n$, $f : I \to \mathbb{R}$ be differentiable and $c \in \mathbb{R}$.

- $\frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$
- $\frac{d}{dt}[c \cdot u(t)] = c \cdot u'(t)$
- $\frac{d}{dt}[f(t) \cdot u(t)] = f'(t) \cdot u(t) + f(t) \cdot u'(t)$
- $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$
- $\frac{d}{dt}[u(f(t))] = f'(t) \cdot u'(f(t))$
- $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$ (only defined for $n = 3$)

Definition *Integrability of vector functions*

$r(t)$ is **integrable** if its components are integrable.
The **integral** of $r(t)$ can be computed by integrating its components.

Definition *Functions of class C^k*

$r : I \to \mathbb{R}^3$ is **continuously differentiable** if r is differentiable and r' is continuous.
If r is continuous it is of class C^0. If r is continuously differentiable k times it is of class C^k.

Definition *Smooth curve*

A curve C is **smooth** if it has a C^1 parametrization $r : I \to \mathbb{R}^n$ with $r'(t) \neq 0 \quad \forall t \in I$

A C^1 parametrization can give a non-smooth curve.

2.3 Arc length

Definition *Length of a curve*

The **length** of a curve C with parametrization $r : [a, b] \to \mathbb{R}^3$ is defined as

$$L = \int_a^b \|r'(t)\| \, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \, dt$$

The length of a curve does not depend on the parametrization.

Definition *Parametrization by arc length*

$s(t) = \int_a^t \|r'(\tau)\| \, d\tau$ is the length of a curve $C = r([a,b])$ between $r(a)$ and $r(t)$
The **parametrization by arc length** is $\widetilde{r}(s) = r(t(s))$ where $t(s)$ is the inverse of $s(t)$.

The tangent vectors of a parametrization by arc length are unit tangent vectors.

Definition *Curvature*

The **curvature** at some point of a smooth curve with C^2 parametrization $r : I \to \mathbb{R}^n$ is
$\kappa = \left\| \frac{dT}{ds} \right\| = \frac{\|T'(t)\|}{\|r'(t)\|}$ where T is the unit tangent vector and s is the arc length.

Definition *Moving frame*

Let C be a smooth curve with a C^3 parametrization $r : I \to \mathbb{R}^3$. The vectors

- $T = \frac{r'(t)}{\|r'(t)\|}$ (**unit tangent vector**)
- $N = \frac{T'(t)}{\|T'(t)\|}$ (**principal normal vector**)
- $B = T \times N$ (**binormal vector**)

are mutually orthogonal and they form a **moving frame**.

Definition *Torsion*

$\tau = -\frac{dB}{ds} \cdot N$

Definition *Osculating plane and circle*

The **osculating plane** spanned by T and N is defined by points P, P_1, P_2 with the limit $P_1, P_2 \to P$.
The **osculating circle** of C at P is the circle in the osculating plane that passes through P
with radius $\frac{1}{\kappa}$ and center a distance $\frac{1}{\kappa}$ from P along the vector N.

Theorem *Frenet-Serret formulas*

$$\frac{d}{ds}T = \kappa N \qquad \frac{d}{ds}N = -\kappa T + \tau B \qquad \frac{d}{ds}B = -\tau N$$

Definition *Generalized definition of the derivative*

$r : I \to \mathbb{R}^n$ is **differentiable** at $t \in I \iff \exists v \in \mathbb{R}^n$ s.t. $\lim_{\tau \to t} \frac{\|r(\tau) - (r(t) + v(\tau - t))\|}{|\tau - t|} = 0.$
where v is the **derivative** of r at t and $L(\tau) = r(t) + v(\tau - t)$ is the **linearization** of r at t.

3 Multivariable functions

When considering functions $D \to \mathbb{R}$, D is an open and connected subset of $\mathbb{R}^n$.

Definition *Level set*

Let $f : D \in \mathbb{R}^n \to \mathbb{R}$ and $k \in \operatorname{range} f$.
Then $\{(x_1, \dots, x_n) \in D \mid f(x_1, \dots, x_n) = k\}$ is called the **level set** of f for k.

3.1 Limits

Definition *Limit of a multivariable function*

L is the **limit** of f at a if $\quad \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $0 < \|x - a\| < \delta \implies |f(x) - L| < \varepsilon$

Definition *Continuity of multivariable functions*

If $a \in D$ and $\lim_{x \to a} f(x) = f(a)$, then f is **continuous** at a.

3.2 Partial derivatives

Definition *Partial derivative w.r.t. x*

Let $D \subseteq \mathbb{R}^2, \quad f : D \to \mathbb{R}, \quad (a,b) \in D, \quad g(x) := f(x,b)$

Suppose that g is differentiable at $x = a$, i.e. the limit $f_x(a,b) = \lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exists.

This limit is called the **partial derivative** of f with respect to x at (a,b).

Definition *Partial derivative w.r.t. y*

Let $D \subseteq \mathbb{R}^2, \quad f : D \to \mathbb{R}, \quad (a,b) \in D, \quad g(x) := f(a,y)$

Suppose that g is differentiable at $y = b$, i.e. the limit $f_y(a,b) = \lim_{h\to 0} \frac{g(h+b)-g(b)}{h}$ exists.

This limit is called the **partial derivative** of f with respect to y at (a,b).

Alternative notations: $f_x(a,b) \equiv \frac{\partial f}{\partial x}(a,b) \equiv \frac{\partial}{\partial x} f(a,b) \equiv \partial_x f(a,b) \equiv D_1 f(a,b) \equiv D_x f(a,b)$

When computing the partial derivative with respect to x, all other variables are seen as constants.

Higher order partial derivatives: $\quad f_{xy}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y,z)\right) \qquad f_{zz}(x,y,z) = \frac{\partial^2 f}{\partial z^2}(x,y,z)$

Definition *Functions of class C^k*

The function $f : D \in \mathbb{R}^n \to \mathbb{R}$ is of class C^k if all partial derivatives of order k exist and are continuous.

Theorem *Clairaut's Theorem (Schwartz's Theorem)*

Let $f : D \in \mathbb{R}^n \to \mathbb{R}$ of class C^2. Then $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f$

Definition *Tangent plane*

Let $f : D \in \mathbb{R}^n \to \mathbb{R}$ of class C^1 and $(a,b) \in D$.
Then $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is called the **tangent plane** of f at $(x,y) = (a,b)$

Definition *Linearization*

The **linearization** of f at a is $L(x) = f(a) + f_{x_1}(a)(x_1 - a_1) + \ldots + f_{x_n}(a)(x_n - a_n)$.

Definition *Differentiability of multivariable functions*

Let $f : D \in \mathbb{R}^n \to \mathbb{R}, a = (a_1, \ldots, a_n) \in D$
f is **differentiable** at a if:

1. The partial derivatives $\frac{\partial f}{\partial x_i}(a)$ exist for all $i \in \{1, \ldots, n\}$.
2. $\lim_{x\to a} \frac{f(x) - L(x)}{\|x-a\|} = 0$

The tangent plane is the graph of the linearization.
A function is differentiable if it can be well approximated by its linearization.

Definition *Gradient*

The **gradient** $\nabla f(a)$ of f at a is given by the vector $\left\langle \frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a) \right\rangle$

Theorem

If $f : D \in \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in D$, then f is continuous at a.

Theorem

Suppose $f : D \in \mathbb{R}^n \to \mathbb{R}$ has continuous partial derivatives at $x = a \in D$.
Then f is differentiable at $x = a$.

The converse of this theorem is not necessarily true.

Theorem *Generalized chain rule*

Define the functions r, f, Z as follows:

- $r : I \subseteq \mathbb{R} \to \mathbb{R}^n, t \mapsto (x_1(t), \ldots, x_n(t))$
- $f : D \subseteq \mathbb{R}^n \to \mathbb{R}, (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$
- $r(I) \subseteq D$
- $Z(t) = (f \circ r)(t)$

Then $\dfrac{\partial Z}{\partial t} = \dfrac{\partial f}{\partial x_1}\dfrac{dx_1}{dt} + \ldots + \dfrac{\partial f}{\partial x_n}\dfrac{dx_n}{dt} = \nabla f(r(t)) \cdot r'(t)$

Theorem *Implicit differentiation*

Let $f(x, y) = 0$ and $\dfrac{\partial f}{\partial y} \neq 0$. Then $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

Let $f(x, y, z) = 0$ and $\dfrac{\partial f}{\partial z} \neq 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{f_x}{f_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{f_y}{f_z}$

Definition *Directional derivative*

Let $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ with $\|u\| = 1$ and $f : D \subseteq \mathbb{R}^n \to \mathbb{R}, x = (x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n)$

Then $D_u f(x) = \lim\limits_{h \to 0} \dfrac{f(x + hu) - f(x)}{h}$ is the **directional derivative** of f in the direction of u.

The existence of all directional derivatives does not necessarily imply differentiability.

Theorem

Suppose f is differentiable. Then $D_u f(x) = \frac{\partial f}{\partial x_1} u_1 + \ldots + \frac{\partial f}{\partial x_n} u_n = \nabla f(x) \cdot u$

Theorem

For f differentiable, $D_u f$ is maximal for $u = \dfrac{\nabla f}{\|\nabla f\|}$ with $\nabla f \neq 0$

Theorem

The gradient of a function is perpendicular to the level set.

Theorem *Implicit function theorem*

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ of class C^1.

Let $a = (a_1, \ldots, a_n)$ such that $a \in \{x = (x_1, \ldots, x_n) | f(x) = c\}$ for some $c \in \mathbb{R}$.

If $\frac{\partial f}{\partial x_n}(a) \neq 0$, then there exist:

- a neighborhood U of $(a_1, \ldots, a_{n_1}) \in \mathbb{R}^{n-1}$
- a neighborhood V of $a_n \in \mathbb{R}$
- a function $g : U \to V$ of class C^1

such that if $(x_1, \ldots, x_{n-1}) \in U$ and $x_n \in V$ satisfy $f(x_1, \ldots, x_{n-1}) = c$ then $x_n = g(x_1, \ldots, x_{n-1})$.

This g is called an **implicit function**. It holds:

$$\frac{\partial g}{\partial x_k}(a_1, \ldots, a_{n-1}) = -\frac{\frac{\partial f}{\partial x_k}(a)}{\frac{\partial f}{\partial x_n}(a)} \qquad k \in (1, \ldots, n-1)$$

Definition *Maximum, minimum, extremum*

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $a \in D$.

- f has a **local maximum** at a if there is a neighborhood U of a such that $f(x) \leq f(a) \quad \forall x \in U$
- f has a **local minimum** at a if there is a neighborhood U of a such that $f(x) \geq f(a) \quad \forall x \in U$
- f has a **global maximum** at a if $f(x) \leq f(a) \quad \forall x \in D$
- f has a **global minimum** at a if $f(x) \geq f(a) \quad \forall x \in D$
- Maxima and minima are called **extrema**.

Theorem

If f has a local extremum at a and f is differentiable at a, then $\nabla f(a) = 0$

Definition *Critical point*

$a \in D \subseteq \mathbb{R}^n$ is called a **critical point** of $f : D \to \mathbb{R}$ if f is differentiable and $\nabla f(a) = 0$

Definition *Saddle*

$f : D \in \mathbb{R}^n \to \mathbb{R}$ has a **saddle** at a if any neighborhood U of a has the following points:

- $x \in U$ with $f(x) > f(a)$
- $y \in U$ with $f(y) < f(a)$

Theorem

Suppose $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ of class C^2 and has a critical point at $(a, b) \in D$.

Let $d = \det \operatorname{Hess} f(a, b)$ where $\operatorname{Hess} f(a, b) := \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$

i.e. $d = f_{xx}(a, b) \cdot f_{yy}(a, b) - (f_{xy}(a, b))^2$ by Schwartz's theorem

- If $d > 0$ and $f_{xx}(a, b) > 0$ then x has a local maximum at (a, b)
- If $d > 0$ and $f_{xx}(a, b) < 0$ then x has a local minimum at (a, b)
- If $d < 0$ then f has a saddle at (a, b)

Theorem *Weierstrass extreme value theorem*

Let $D \in \mathbb{R}^n$ compact and $f : D \to \mathbb{R}$ continuous.
Then there exist $x, y \in D$ such that x is a global maximum of f and y is a global minimum of f.

Theorem *Lagrange multiplier method*

Let $D \subseteq \mathbb{R}^n$ and $f, g : D \to \mathbb{R}$. Let $S = \{x \in D \mid g(x) = c\}$ for a fixed c in the range of g.
If f restricted to S has an extremum at $a \in S$, then there exists a **Lagrange multiplier** $\lambda \in \mathbb{R}$ such that $\nabla f(a) = \lambda \nabla g(a)$

Theorem *Lagrange multiplier method with two constraints*

Let $f, g, h : D \in \mathbb{R}^n \to \mathbb{R}$. Let $S = \{x \in D \mid g(x) = c \text{ and } h(x) = d\}$.
Then if f constrained to S has a minimum at $a \in S$, there exist $\lambda, \mu \in \mathbb{R}$ such that $\nabla f(a) = \lambda \nabla g(a) + \mu \nabla h(a)$

3.3 Double integrals

Definition *Double Riemann integral*

The definition of the Riemann intergral over $\mathbb{R}^2$ is analogous to the definition for $\mathbb{R}$:

$$\iint_R f \, \mathrm{dA} = \lim_{\Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(c_{ij}) \Delta x_i \Delta y_j$$

where c_{ij} is a **sample point** in each rectangle. If this limit exists, f is **Riemann integrable**.

Theorem

If f is continuous on R then f is integrable on R.

Theorem *Fubini's theorem*

Suppose f is continuous on R. Then its integral is equivalent to two **iterated integrals**:

$$\iint_R f \, \mathrm{dA} = \int_a^b \int_c^d f(x, y) \, \mathrm{dy} \, \mathrm{dx} = \int_c^d \int_a^b f(x, y) \, \mathrm{dx} \, \mathrm{dy}$$

Definition *Double integral with arbitrary domain*

To define $\iint_D f\,dA$ where D is bounded, let R be any rectangle containing D.

Extend f to R by defining $f^{\text{ext}}(x,y) = \begin{cases} f(x,y) \text{ if } (x,y) \in D \\ 0 \text{ if } (x,y) \notin D \end{cases}$

We define $\iint_D f\,\mathrm{dA}$ to be $\iint_R f^{\text{ext}}\,\mathrm{dA}$

Definition *Elementary regions in* $\mathbb{R}^2$

A region in $\mathbb{R}^2$ is of:

- **type 1** if it is bounded by two functions $\delta(x)$ and $\gamma(x)$
- **type 2** if it is bounded by two functions $\alpha(y)$ and $\beta(y)$
- **type 3** if it is of type 1 and type 2

Theorem

Suppose f is continuous on D.

- If D is of type 1, then $\iint\limits_D f\,\mathrm{dA} = \int\limits_a^b \int\limits_{\gamma(x)}^{\delta(x)} f(x,y)\,\mathrm{dy}\,\mathrm{dx}$
- If D is of type 2, then $\iint\limits_D f\,\mathrm{dA} = \int\limits_c^d \int\limits_{\beta(y)}^{\alpha(y)} f(x,y)\,\mathrm{dx}\,\mathrm{dy}$

Definition *Polar coordinates*

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x}$$

When integrating using polar coordinates each "subrectangle" is bounded by circles and rays.

Definition *Double integral with polar coordinates*

$$\iint\limits_D f(x,y)\,\mathrm{dA} = \iint f(r\cos\theta, r\sin\theta) r\,\mathrm{dr}\,\mathrm{d}\theta$$

Definition $\mathbb{R}^2$ *Jacobian matrix*

Let $T : (x,y) \mapsto (x(u,v), y(u,v))$. Then its **Jacobian** is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Theorem *Substitution rule for double integrals*

Let $T : (x,y) \mapsto (x(u,v), y(u,v))$ be a bijective C^1 map from $D \in \mathbb{R}^2$ to $D^* \in \mathbb{R}^2$. Then,

$$\iint\limits_D f(x,y)\,\mathrm{dx}\,\mathrm{dy} = \iint\limits_{D^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{du}\,\mathrm{dv}$$

3.4 Triple integrals

Definition *Triple Riemann integral*

The **triple integral** of f over a box B is

$$\iiint_B f\,\mathrm{dV} = \lim_{\Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(c_{ijk}) \Delta x_i \Delta y_j \Delta z_k$$

provided that the limit exists. If it does, f is **Riemann integrable** over B.

Theorem *Fubini's theorem for triple integrals*

Suppose f is continuous on f. Then

$$\iiint_B f\,\mathrm{dV} = \int_a^b \int_c^d \int_p^q f(x,y,z)\,\mathrm{dz}\,\mathrm{dy}\,\mathrm{dx}$$

or any of the 5 other orders.

Definition *Elementary regions in* $\mathbb{R}^3$

A region in $\mathbb{R}^2$ is of:

- **type 1** if it is bounded by two functions $\psi(x,y)$ and $\varphi(x,y)$
- **type 2** if it is bounded by two functions $\alpha(y,z)$ and $\beta(y,z)$
- **type 3** if it is bounded by two functions $\gamma(x,z)$ and $\delta(x,z)$
- **type 4** if it is of type 1, 2 and 3

Theorem

Suppose f is continuous on W.
Let S be the **shadow** of W on the (x,y), (y,z) and (x,z) plane respectively.

- If W is of type 1, then $\iiint_W f\,\mathrm{dV} = \iint_S \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,\mathrm{dz}\,\mathrm{dA}$
- If W is of type 2, then $\iiint_W f\,\mathrm{dV} = \iint_S \int_{\beta(y,z)}^{\alpha(y,z)} f(x,y,z)\,\mathrm{dx}\,\mathrm{dA}$
- If W is of type 3, then $\iiint_W f\,\mathrm{dV} = \iint_S \int_{\delta(x,z)}^{\gamma(x,z)} f(x,y,z)\,\mathrm{dy}\,\mathrm{dA}$

Definition *Cylindrical coordinates*

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2+y^2} \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Definition *Spherical coordinates*

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi \qquad \rho = \sqrt{x^2+y^2+z^2}$$

$$\tan\theta = \frac{y}{x} \qquad \tan\phi = \frac{\sqrt{x^2+y^2}}{z} \qquad \cos\phi = \frac{z}{\rho}$$

θ is the **azimuthal angle**, ϕ is the **polar angle**.

Definition $\mathbb{R}^3$ *Jacobian matrix*

Let $T : (x, y, z) \mapsto (x(u, v, w), y(u, v, w), z(u, v, w))$. Then its **Jacobian** is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Theorem *Substitution rule for triple integrals*

Let $T : (x, y, z) \mapsto (x(u, v, w), y(u, v, w), z(u, v, w))$ be a bijective C^1 map from $W \in \mathbb{R}^3$ to $W^* \in \mathbb{R}^3$. Then,

$$\iiint\limits_W f(x, y, z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \iiint\limits_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w$$

4 Vector fields

Definition *Vector field*

A **vector field** is a function F that assigns a vector $F(x) \in \mathbb{R}^n$ to each point $x \in \mathbb{R}^n$.

Definition *Gradient vector field*

A vector field $F : D \in \mathbb{R}^n \to \mathbb{R}^n$ is **conservative** or a **gradient vector field** if there exists a function $f : D \to \mathbb{R}$ with $F = \nabla f$, where f is the **potential function** for F.

4.1 Line integrals

Definition *Line integral*

Let C be a smooth curve in $\mathbb{R}^n$ with parametrization $r : [a, b] \to \mathbb{R}^n, t \mapsto r(t), r([a, b]) = C$
Then we can define two types of integrals:

1. Let $f : D \in \mathbb{R}^n \to \mathbb{R}$ be a continuous scalar-valued function and $C \subset D$.
   Then the **line integral** of f along C is defined as $\displaystyle\int_C f \,\mathrm{d}s = \int_a^b f(r(t)) \|r'(t)\| \,\mathrm{d}t$
2. Let $F : D \in \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field and $C \subset D$.
   Then the **line integral** of F along C is defined as $\displaystyle\int_C F \cdot \mathrm{d}r = \int_b^a F(r(t)) \cdot r'(t) \,\mathrm{d}t$

Theorem

The line integral of a scalar-valued function does not depend on the parametrization.

Definition *Piecewise smooth curve*

A curve C is **piecewise smooth** if C is a concatenation of smooth curves $C_1, \ldots, C_n$ where the intitial point of C_{k+1} agrees with the final point of C_k $\forall k \in \{1, \ldots, n-1\}$

Theorem

If C is a piecewise smooth curve, then $\displaystyle\int_C f \,\mathrm{d}s = \sum_{k=1}^n \int_{C_k} f \,\mathrm{d}s$

Differential form notation: If $n = 3$, then $F = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ and

$$\int F \cdot r' \,\mathrm{d}t = \int (Px' + Qy' + Rz') \,\mathrm{d}t = \int P \,\mathrm{d}x + Q \,\mathrm{d}y + R \,\mathrm{d}z$$

Theorem

Let F be a continuous vector field on $D \in \mathbb{R}^n$.
Let $r : [a, b] \to \mathbb{R}^n$ and $\widetilde{r} : [c, d] \to \mathbb{R}^n$ be parametrizations of the same smooth curve $C \subset D$. Then

$$\int_C F \cdot \mathrm{dr} = \begin{cases} \int_C F \cdot \mathrm{d}\widetilde{\mathrm{r}} \text{ if } r(a) = \widetilde{r}(c) \text{ and } r(b) = \widetilde{r}(d) \\ -\int_C F \cdot \mathrm{d}\widetilde{\mathrm{r}} \text{ otherwise} \end{cases}$$

Definition *Oriented curve*

A curve C is called an **oriented curve** it has a fixed orientation (direction of unit tangent vectors)
Notation: $-C$ is the same curve as C but with the opposite orientation.

If C is an oriented curve, then $\int_C F \cdot \mathrm{dr} = -\int_{-C} F \cdot \mathrm{dr}$

Line integrals of vector fields on oriented curves do not depend on the parametrization.

Theorem *Fundamental theorem of line integrals*

Let C be a smooth curve with parametrization $r : [a, b] \to \mathbb{R}^n$ and let F be a conservative vector field with potential function f. Suppose that F is continuous.
Then: $\int_C F \cdot \mathrm{dr} = f(r(b)) - f(r(a))$

Definition *Independent of path*

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field.
F is **independent of path** if $\int_C F \cdot \mathrm{ds} = \int_{\widetilde{C}} F \cdot \mathrm{ds}$ for all $\widetilde{C}$ with the same endpoints as C.

Theorem

Conservative vector fields are independent of path.

Definition *Closed curve*

A **closed curve** is a curve where the initial and final point agree.

Theorem

Let F be a continuous vector field. Then:

$$F \text{ is independent of path} \iff \int F \cdot \mathrm{ds} = 0 \text{ for all closed curves } C$$

Definition *Connected domain*

D is called **connected** if any two points in D can be joined by a curve in D.

Theorem

If $D \in \mathbb{R}^n$ connected and $F : D \to \mathbb{R}^n$ independent of path, then F is conservative.

Definition *Simply connected domain*

D is called **simply connected** if it is connected and every closed curve in D can be contracted to a single point without leaving D.

Theorem

Let $F = Pi + Qj$ be a vector field in a simply connected $D \subset \mathbb{R}^2$ with P and Q being C^1. Then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \iff F \text{ conservative}$$

Definition *Simple curve*

A curve in the plane is called **simple** if it has not self-intersections

Theorem *Green's theorem*

Let D be a closed and bounded region in $\mathbb{R}^2$ whose boundary consists of finitely many simple, closed and piecewise C^1 curves. Orient ∂D such that D is on the left as one traverses ∂D. Then

$$\oint_{\partial D} F \cdot \mathrm{ds} = \oint_{\partial D} P \,\mathrm{dx} + Q \,\mathrm{dy} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{dx}\,\mathrm{dy}$$

4.2 Divergence and curl

Definition *Del operator*

$$\text{On } \mathbb{R}^3\text{: } \nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

$$\text{In general: } \nabla = \boldsymbol{e}_1 \frac{\partial}{\partial x_1}, \boldsymbol{e}_2 \frac{\partial}{\partial x_2}, \dots, \boldsymbol{e}_n \frac{\partial}{\partial x_n} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)$$

The gradient $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$ is defined using the del operator.

∇ turns a scalar field into a vector field.
The direction of ∇f is that of greatest increase of f.
The magnitude of ∇f is the rate of maximum increase.

Definition *Divergence*

The **divergence** of F, denoted $\operatorname{div} F$ or $\nabla \cdot F$ is defined using the del operator:

$$\nabla \cdot F = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right) \cdot (F_1, F_2, \dots, F_n) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \dots + \frac{\partial F_n}{\partial x_n}$$

$\cdot\nabla$ turns a vector field into a scalar field.

Definition *Curl*

Suppose that $F(x, y, z)$ is a vector field on $X \subseteq \mathbb{R}^3$ only.
The **curl** of F, denoted $\operatorname{curl} F$ or $\nabla \times F$ is defined using the del operator:

$$\nabla \times F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (P, Q, R) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \boldsymbol{k}$$

$\times\nabla$ turns a vector field into another vector field.
The direction of $\nabla \times F(x)$ is the orientation (via a right hand rule) of the local rotation of F at x.
The magnitude of $\nabla \times F(x)$ is the rate of this local rotation.

Theorem

If f is a scalar-valued function of class C^2, then $\quad \nabla \times \nabla f = 0$

Theorem

If F is a vector field of class C^2 on $D \subseteq \mathbb{R}^3$, then $\quad \nabla \cdot (\nabla \times f) = 0$

Theorem

Let $F : D \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field with D simply connected.

$$F \text{ conservative} \iff \operatorname{curl} F = 0$$

4.3 Parametric surfaces

Definition *Parametrized surface*

Let D be a region in $\mathbb{R}^2$ consisting of a connected open set, possibly together with some (or all) of its boundary points.
A **parametrized surface** in R^3 is a continuous map $X : D \to R^3$ that is injective on D, except possibly along ∂D

Definition *Coordinate curves*

Let S be a surface parametrized by $X : D \to \mathbb{R}^3$.

- The s-coordinate curve is the image of the map $s \mapsto X(s, t_0)$.
- The t-coordinate curve is the image of the map $t \mapsto X(s_0, t)$.

Definition *Tangent vectors of coordinate curves*

If $X(s,t) = (x(s,t), y(s,t), z(s,t))$ is differentiable at $(s_0, t_0) \in D$, then a tangent vector $T_s(s_0, t_0)$ to the s-coordinate curve $X(s_0, t)$ at (s_0, t_0) may be computed as

$$T_s(s_0,t_0) = \frac{\partial \boldsymbol{X}}{\partial s}(s_0,t_0) = \frac{\partial x}{\partial s}(s_0,t_0)\boldsymbol{i} + \frac{\partial y}{\partial s}(s_0,t_0)\boldsymbol{j} + \frac{\partial z}{\partial s}(s_0,t_0)\boldsymbol{k}$$

Similarly, a tangent vector $T_t(s_0, t_0)$ to the t-coordinate curve $X(s, t_0)$ at (s_0, t_0) is given by

$$T_t(s_0,t_0) = \frac{\partial \boldsymbol{X}}{\partial t}(s_0,t_0) = \frac{\partial x}{\partial t}(s_0,t_0)\boldsymbol{i} + \frac{\partial y}{\partial t}(s_0,t_0)\boldsymbol{j} + \frac{\partial z}{\partial t}(s_0,t_0)\boldsymbol{k}$$

Definition *Smooth parametrized surface*

$S = X(D)$ is **smooth** at $X(s_0, t_0)$ if X is of class C_1 in a neighborhood of $(s_0, t_0) \in D$ and if $T_s(s_0, t_0) \times T_t(s_0, t_0) \neq 0$.
If S is smooth at every point $X(s_0, t_0)$, then we call it a **smooth parametrized surface**.

Definition *Standard normal vector*

The nonzero vector $N(s_0, t_0) = T_s(s_0, t_0) \times T_t(s_0, t_0)$ is the **standard normal vector** given by the parametrization.

Definition *Piecewise smooth parametrized surface*

A **piecewise smooth parametrized surface** is the union of images of finitely many parametrized surfaces $X_i : D_i \to \mathbb{R}_3$, $i = 1, \ldots, m$, such that

- Each X_i is injective on D_i, except possibly along ∂D_i
- Each $S_i = X_i(D_i)$ is smooth, except possibly at finitely many points.

Definition *Area of a smooth parametrized surface*

Suppose $S = X(D)$ is a smooth parametrized surface. The **surface area** of S is given by

$$\iint_D \|T_s \times T_t\| \,\mathrm{ds}\,\mathrm{dt} = \iint_D \|N(s,t)\| \,\mathrm{ds}\,\mathrm{dt}$$

4.4 Surface integrals

Definition *Scalar surface integral*

Let:

- $X : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset R^2$ is a bounded region;
- $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a continuous function, with X containing $S = X(D)$.

The **scalar surface integral** of f along X, denoted $\iint_X f \, \mathrm{dS}$, is

$$\iint_X f \, \mathrm{dS} = \iint_D f(X(s,t)) \|T_s \times T_t\| \, \mathrm{ds} \, \mathrm{dt} = \iint_D f(X(s,t)) \|N(s,t)\| \, \mathrm{ds} \, \mathrm{dt}$$

Definition *Vector surface integral*

Let:

- $X : D \to \mathbb{R}^3$ be a smooth parametrized surface, where $D \subset R^2$ is a bounded region;
- $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous vector field, with X containing $S = X(D)$.

The **vector surface integral** (or **flux**) of F along X, denoted $\iint_X F \cdot \mathrm{dS}$, is

$$\iint_X F \cdot \mathrm{dS} = \iint_D F(X(s,t)) \cdot N(s,t) \, \mathrm{ds} \, \mathrm{dt}$$

Definition *Unit normal vector*

If $X : D \to \mathbb{R}^3$ is a smooth parametrized surface then we can define the **unit normal vector** $n(s,t) = \dfrac{N(s,t)}{\|N(s,t)\|}$

Theorem

$$\iint_X F \cdot \mathrm{dS} = \iint_X (F \cdot n) \, \mathrm{dS}$$

Theorem

Scalar surface integrals do not depend on the parametrization:
if Y is any smooth reparametrization of X, then $\displaystyle\iint_Y f \, \mathrm{dS} = \iint_X f \, \mathrm{dS}$

Definition *Orientability*

A smooth, connected surface S is **orientable** (or **two-sided**) if a single unit normal vector can be defined at each point of S so that the collection of these vectors varies continuously over S.
Otherwise, S is called **nonorientable**.

Definition *Oriented surface*

If S is orientable, then it has two orientations. A smooth surface together with a choice of continuous orientation normal vectors is called an **oriented surface**.

Theorem

If Y is a smooth reparametrization of X, then,

$$\iint_Y F \cdot \mathrm{dS} = \begin{cases} \iint_X F \cdot \mathrm{dS} \text{ if } Y \text{ is orientation-preserving} \\ -\iint_X F \cdot \mathrm{dS} \text{ if } Y \text{ is orientation-reversing} \end{cases}$$

Definition *Generalization to piecewise smooth surfaces*

Let S be a piecewise smooth connected surface. In particular, suppose that $S = S_1 \cup \ldots \cup S_k$ where each S_i is smooth. Then we define the **scalar surface integral** of a function f over S by

$$\iint_S f \, \mathrm{dS} = \iint_{S_1} f \, \mathrm{dS} + \ldots + \iint_{S_k} f \, \mathrm{dS}$$

If each S_i is oriented so that S is oriented, we define the **vector surface integral** of a vector field F along S as

$$\iint_S F \cdot \mathrm{dS} = \iint_{S_1} F \cdot \mathrm{dS} + \ldots + \iint_{S_k} F \cdot \mathrm{dS}$$

Definition *Induced orientation*

Let S be a bounded, piecewise smooth surface region in $\mathbb{R}^3$, oriented by unit normal n at each point.
Let C' be and simple closed curve lying in S.
The normal n can be used to orient C' by a right-hand rule.
We call this orientation of C' the one **induced** from the orientation of S,
and we say that C' with the induced orientation is **oriented consistently** with S.

Theorem *Stokes' theorem*

Suppose that:

- S is a bounded, piecewise smooth, oriented surface in $\mathbb{R}^3$, oriented by unit normal n at each point.
- ∂S consists of finitely many piecewise C^1 simple closed curves, each oriented consistently with S.
- F is a vector field of class C^1 whose domain includes S.

Then,

$$\iint_S \nabla \times F \cdot \mathrm{dS} = \oint_{\partial S} F \cdot \mathrm{dS}$$

Stokes' theorem also implies that

$$\iint_S \nabla \times F \cdot \mathrm{dS} = \iint_{S^*} \nabla \times F \cdot \mathrm{dS}$$

for any S^* with the same boundaries as S.

Theorem *Gauss' divergence theorem*

Let D be a bounded solid region in $\mathbb{R}^3$. Suppose that the boundary of D consists of finitely many smooth orientable parametrized surfaces, each oriented by a unit normal vector pointing to the outside of D. For a C^1 vector field $F : D \subset \mathbb{R}^3 \to \mathbb{R}^3$, it holds:

$$\iiint_D \nabla \cdot F \, \mathrm{dV} = \oiint_{\partial D} F \cdot \mathrm{dS}$$